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w^*-Algebra, Poincaré Group, and Quantum Kinetic Theory

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The w^*-algebra in the standard representation is used to define a vector space for representations of Lie algebras. The Poincaré group is studied as in thermofield dynamics (TFD) with the result that the notion of phase space is introduced from the structure of the Poincaré–Lie algebra. The basis of quantum-field kinetic theory is analyzed in association with TFD. As a particular case, the Jüttner distribution is derived.

Recently, several physical and algebraic aspects of thermal theories have been studied through the formalism known as thermofield dynamics (TFD), proposed by Takahashi and Umezawa (1975) and developed, for instance, by Umezawa *et al.* (1982) and Umezawa (1993) as an operator approach to treat thermal phenomena. TFD is defined via two ingredients: a Bogoliubov transformation, introducing thermal effects via a vacuum correlation, and a doubling in the dynamical variables.

Such a TFD structure was first associated with c^*-algebra by Ojima (1981), motived by the derivation of the so-called KMS (Kubo, Martin, Schwinger) equilibrium conditions. On the other side, different aspects of (thermal) symmetries have been analyzed and explored as a consequence of the operator nature of TFD. As an instance, Umezawa (1993) and Chu and Umezawa (1994) showed that the generators of the thermal Bogoliubov

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transformation are associated with $su(1,1)$ for bosons and $su(2)$ for fermions. Furthermore, Celeghini *et al.* (1991), Song *et al.* (1993), Ioro and Vitiello (1994), Srivastava *et al.* (1995), and Vitiello (1996) explored elements of q-groups in connection with the notion of the TFD-dual variables, and the effect of temperature has been studied as deformations in the Weyl–Heisenberg algebra. In this scenario of q-groups, Kopf *et al.* (1997) pointed out that the notion of a bialgebra is a structural constituent for thermal theories.

Using the TFD Hilbert space as the space of representation for Lie groups, we have analyzed the Galilei and Poincaré symmetries (see, e.g., Santana and Khanna, 1995; Santana *et al.*, 1996; da Silva *et al.*, 1997). As a result the TFD basic equations have been derived through a study of Lie algebras; a physical interpretation of the dual variables arising in thermal field theories has been introduced; a definition of a Klein–Gordon-like Liouville von Neumann equation has been deduced; and the development of the classical counterpart of TFD has been proposed. Starting from the notion of w^*-algebras, we have shown (see, e.g., Neto *et al.*, 1996) how this Lie-algebra method (we call it here *-Lie algebra) can formally emerge in thermal theories, and in this context the phase-space Wigner function has been derived as a matter of representations of the Galilei symmetries. So this approach can avoid the common difficulties with the available formalisms used to introduce the concept of Wigner functions in general situations, as pointed out by de Groot *et al.* (1980) and Cabo and Shabad (1987) in connection with the nontrivial generalization of the Weyl transformation to deal with relativistic systems. Beyond that, the *-Lie algebra formalism can be seen as an alternate generalization to the studies of Grelland (1984, 1993), based on the standard representation of w^*-algebras, treating the classical limit of relativistic quantum systems and reducible Dirac representations for the classical and quantum theory. One benefit of such an approach has been a profitable way to study quantum stochasticity, as set forth by Benatti *et al.* (1991) and Prigogine *et al.* (1990), and as emphasized by Grelland (1993).

In this paper, the preliminary results presented by Santana and Khanna (1995) and Neto *et al.* (1996) are developed, by introducing the relativistic quantum-kinetic theory based on the analysis of representations of the *-Poincaré algebra and in association with the TFD formalism. In the steady thermal case, in particular, we derive the Jüttner distribution, and we show how the concept of relativistic quantum phase space arises from an analysis of the representations of Poincaré symmetries, with two (nonordinary) remarkable aspects: simplicity and covariance. To proceed further, we first introduce the basic elements of representations of *-Lie algebras.

Let $(\mathcal{H}, \pi(\mathcal{A}))$ be a faithful realization of $\mathcal{A}$, a w^*-algebra, where $\mathcal{H}$ is a Hilbert space. $\pi(\mathcal{A})$: $\mathcal{H} \to \mathcal{H}$ is, then a *-isomorphism of $\mathcal{A}$ by linear operators in $\mathcal{H}$. Taking $|\xi\rangle \in \mathcal{H}$ to be normalized, it follows that $\langle\xi|\pi(A)|\xi\rangle$, for every $A \in \mathcal{A}$, defines a state over $\mathcal{A}$ denoted by $\omega_\xi(A) = \langle \xi | \pi(A) | \xi \rangle$. Such states are called *vector states*. As shown by Gelfand, Naimark, and Segal (GNS), the inverse is also true, i.e., every state ω of a w^*-algebra $\mathcal{A}$ admits a vector representation $|\xi_\omega\rangle \in \mathcal{H}$ such that $w(A) \equiv \langle \xi_\omega | \pi_\omega(A) | \xi_\omega \rangle$. This realization is called the *GNS construction* (see, e.g., Takesaki, 1970; Emch, 1972; Bratteli and Robison, 1979). In order to emphasize the dependence of the representation space and of the operators on the state ω in $\mathcal{A}$, it is usual to denote such a realization by $(\mathcal{H}_\omega, \pi_\omega(\mathcal{A}))$.

The standard (Tomita–Takesaki) representation is a class of representations defined as follows. Let σ: $\mathcal{H}_\omega \to \mathcal{H}_\omega$ be a conjugation in $\mathcal{H}_\omega$, that is, σ is an antilinear isometry such that $\sigma^2 = 1$. $(\mathcal{H}_\omega, \pi_\omega(\mathcal{A}))$ is a Tomita–Takesaki representation of the w^*-algebra $\mathcal{A}$ iff $\sigma\pi_\omega(\mathcal{A})\sigma = \tilde{\pi}_\omega(\mathcal{A})$ defines a *-antiisomorphism on the linear operators. It follows that $(\mathcal{H}_\omega, \tilde{\pi}_\omega(\mathcal{A}))$ is a faithful antirealization of $\mathcal{A}$. Notice that $\tilde{\pi}_\omega(\mathcal{A})$ is the commutant of $\pi_\omega(\mathcal{A})$, that is, $[\pi_\omega(\mathcal{A}), \tilde{\pi}_\omega(\mathcal{A})| = 0$. In this representation, the representative vectors of the states are invariant under σ, that is, $\sigma|\xi_\omega\rangle = |\xi_\omega\rangle$. For the sake of simplicity, the elements of $\pi_\omega(\mathcal{A})$ will be denoted by A, and those of $\tilde{\pi}_\omega(\mathcal{A})$ by $\tilde{A}$.

The properties of the *-automorphisms in the w^*-algebra $\mathcal{A}$ can be defined through unitary operators, say $U(\tau)$, such that $[U(\tau), \sigma] = 0$. Since unitary operators $U(\tau)$ can be written as $U(\tau) = \exp(i\tau\hat{A})$, where $\hat{A}$ is a transformation (symmetry) generator, and considering that $U(\tau)$ commutes with σ, it follows that $\sigma\hat{A}\sigma = -\hat{A}$. Therefore, $\hat{A}$ can be written as an odd polynomial function of $A - \tilde{A}$, i.e.,

$$\hat{A} = f(A - \tilde{A}) = \sum_{n=0}^{\infty} (A - \tilde{A})^{2n+1} \tag{1}$$

Now let us consider $\mathcal{H}_\omega$ as the space of representations for Lie algebras. First, we observe that the two classes of elements of $\mathcal{H}_\omega$, A and $\tilde{A}$, can be replaced, without loss of generality, by the two classes of operators A and $\hat{A}$. The hat operators naturally describe symmetries. However, we need to describe the role played by the A (no-hat) operators. In the context of space–time groups, we can take the no-hat operators as the observables describing the properties of the dynamical system.

Let $l = \{a_i, i = 1, 2, 3, \ldots\}$ be a Lie algebra over the (real) field $\mathbf{R}$, of a Lie group $\mathcal{L}$, characterized by the algebraic relations $a_i \diamond a_j = C_{ijk} a_k$, where $C_{ijk} \in \mathbf{R}$ are the structure constants and $\diamond$ is the Lie product. Considering the Tomita–Takesaki space $\mathcal{H}_\omega$ as the space of representations for l, then we can write

$$[\hat{A}_i, \hat{A}_j] = iC_{ijk}\hat{A}_k \tag{2}$$

We have, however, ancillary commutation relations, specifying (i) the way the generators can change the observables, and (*ii*) the Abelian (or non-Abelian) nature of the observables in regard to the measurement process. Conditions (*i*) and (*ii*) respectively, can be expressed in general by

$$[\hat{A}_i, A_j] = iD_{ijk}A_k \tag{3}$$

$$[A_i, A_j] = iE_{ijk}A_k \tag{4}$$

By doing so, we split the twofold structure of $\mathcal{H}_\omega$, for studying a Lie symmetry. The resulting algebra based on Eqs. (2)–(4) will be referred to here as a *-Lie algebra, and will be denoted by *-*l*.

Some aspects of *-*l* are worthy of comment. (*a*) To each generator $\hat{A}_i$ there is an associated observable A_i. This is defined by the algebraic nature of the standard representation, and is also compatible with the physics. (*b*) If $C_{ijk} = D_{ijk} = E_{ijk}$, then we have a reducible Tomita–Takesaki structure as studied by Santana and Khanna (1995) and Neto *et al.* (1996). (*c*) If $E_{ijk} = 0$, then the representation can describe a classical system, since by construction, all the observables commute with each other, as developed by Santana *et al.* (1996) and da Silva *et al.* (1997).

Here we explore the simplest situation for hat-variables in Eq. (1), in which we define

$$\hat{A} = A - \tilde{A} \tag{5}$$

That is, $\hat{A}$ is an element of the set $\hat{\pi}_\omega(\mathcal{A}) = \pi_\omega(\mathcal{A}) - \hat{\pi}_\omega(\mathcal{A})$. Let us then apply this representation to study the *-Poincaré algebra *-*p*, which is given by the following commutation relations:

$$[M_{\mu\nu}, P_\sigma] = i(g_{\nu\sigma}P_\mu - g_{\sigma\mu}P_\nu) \tag{6}$$

$$[P_\mu, P_\nu] = 0 \tag{7}$$

$$[M_{\mu\nu}, M_{\sigma\rho}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}) \tag{8}$$

$$[\hat{M}_{\mu\nu}, P_\sigma] = [M_{\mu\nu}, \hat{P}_\sigma] = i(g_{\nu\sigma}P_\mu - g_{\sigma\mu}P_\nu) \tag{9}$$

$$[\hat{P}_\mu, P_\nu] = 0 \tag{10}$$

$$[\hat{M}_{\mu\nu}, M_{\sigma\rho}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}) \tag{11}$$

$$[\hat{M}_{\mu\nu}, \hat{P}_\sigma] = i(g_{\nu\sigma}\hat{P}_\mu - g_{\sigma\mu}\hat{P}_\nu) \tag{12}$$

$$[\hat{P}_\mu, \hat{P}_\nu] = 0 \tag{13}$$

$$[\hat{M}_{\mu\nu}, \hat{M}_{\sigma\rho}] = -i(g_{\mu\rho}\hat{M}_{\nu\sigma} - g_{\nu\rho}\hat{M}_{\mu\sigma} + g_{\mu\sigma}\hat{M}_{\rho\nu} - g_{\nu\sigma}\hat{M}_{\rho\mu}) \tag{14}$$

where $M_{\mu\nu}$ stands for rotations and P_μ for translations. The metric tensor is such that $diag(g_{\mu\nu}) = (1, -1, -1, -1)$, and $g_{\mu\nu} = 0$ for $\mu \neq \nu$; $\mu, \nu = 0, 1, 2, 3$. This algebra can be written in a shorthand notation as

$$
\begin{aligned}
[\mathbf{M}, \mathbf{P}] &= i\mathbf{P}, & [\hat{\mathbf{M}}, \mathbf{M}] &= i\mathbf{M} \\
[\mathbf{P}, \mathbf{P}] &= 0, & [\hat{\mathbf{M}}, \hat{\mathbf{P}}] &= i\hat{\mathbf{P}} \\
[\mathbf{M}, \mathbf{M}] &= i\mathbf{M}, & [\hat{\mathbf{M}}, \hat{\mathbf{M}}] &= i\mathbf{M} \\
[\hat{\mathbf{M}}, \mathbf{P}] &= i\mathbf{P}, & [\hat{\mathbf{P}}, \hat{\mathbf{P}}] &= 0 \\
[\hat{\mathbf{P}}, \mathbf{P}] &= 0
\end{aligned}
$$

Defining the tilde variables $\tilde{\mathbf{P}} = \mathbf{P} - \hat{\mathbf{P}}$ and $\tilde{\mathbf{M}} = \mathbf{M} - \hat{\mathbf{M}}$ by using Eq. (5), we have for the nonnull commutation relations

$$
\begin{aligned}
[\mathbf{M}, \mathbf{P}] &= i\mathbf{P} \\
[\mathbf{M}, \mathbf{M}] &= i\mathbf{M} \\
[\tilde{\mathbf{M}}, \tilde{\mathbf{P}}] &= -i\tilde{\mathbf{P}} \\
[\tilde{\mathbf{M}}, \tilde{\mathbf{M}}] &= -i\tilde{\mathbf{M}}
\end{aligned}
$$

Then, introducing the Pauli–Lubanski matrices as usual,

$$w_\mu = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} M^{\nu\sigma} P^\rho$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol, we have the invariants of $*$-**p**

$$W = w_\mu w^\mu \tag{15}$$

$$P^2 = P_\mu P^\mu \tag{16}$$

$$\hat{W} = 2\hat{w}_\mu w^\mu - \hat{w}_\mu \hat{w}^\mu \tag{17}$$

$$\hat{\Pi} = 2\hat{P}_\mu P^\mu - \hat{P}_\mu \hat{P}^\mu \tag{18}$$

where

$$\hat{w}_\mu = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\hat{M}^{\nu\sigma}P^\rho + \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}M^{\nu\sigma}\hat{P}^\rho - \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\hat{M}^{\nu\sigma}\hat{P}^\rho$$

Notice that the vector

$$\overline{w}_\mu = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\hat{M}^{\nu\sigma}\hat{P}^\rho$$

can be used to define the scalar $\overline{W} = \overline{w}_\mu \overline{w}^\mu$, which is not an invariant of $*$-p, but rather of the subalgebra of $*$-p given by Eqs. (12)–(14). Using the definition of the hat variables, Eq. (5), we find

$$\begin{aligned}\hat{W} &= (w_\mu w^\mu)^\wedge \\ &= w_\mu w^\mu - (w_\mu w^\mu)^\sim \\ &= w_\mu w^\mu - \tilde{w}_\mu \tilde{w}^\mu \end{aligned} \tag{19}$$

and, in the same way,

$$\hat{\Pi} = P_\mu P^\mu - \tilde{P}_\mu \tilde{P}^\mu \tag{20}$$

Representations for *-p can be built taking the usual representations for the Poincaré group as a starting point. In this case, we can write explicitly for the nontilde variables

$$[(P^2 - m^2)]^l |\Psi\rangle = [(P^2 - m^2) \otimes 1] |\Psi\rangle = 0 \tag{21}$$

and for the tilde variables

$$[(P^2 - m^2)]^r |\Psi\rangle = 1 \otimes [(P^2 - m^2)] |\Psi\rangle = 0$$

where we have used the following convenient notation for (nontilde and tilde) operators acting on the state $|\Psi\rangle$:

$$O = O^l |\Psi\rangle = [O \otimes 1] |\Psi\rangle \quad \text{and} \quad \tilde{O} = O^r |\Psi\rangle = [1 \otimes O] |\Psi\rangle$$

Considering

$$|\Psi\rangle = |\phi\rangle \otimes \langle\phi| = |\phi\rangle \otimes |\tilde{\phi}\rangle \tag{22}$$

with $\langle\phi|\phi\rangle = 1$, and

$$(P^2 - m^2)|\phi\rangle = (\Box - m^2)|\phi\rangle = 0 \tag{23}$$

then we have

$$\begin{aligned}\Box^l |\Psi\rangle &= (\otimes 1)|\Psi\rangle \\ &= (|\phi\rangle)\langle\phi| \\ \langle\Box^r |\Psi\rangle &= (1\otimes)|\Psi\rangle \\ &= |\phi\rangle(\langle\phi|)\end{aligned}$$

From these results, it follows that

$$[\Box^l - \Box^r]|\Psi\rangle = \hat{\Pi}|\Psi\rangle = 0 \tag{24}$$

Taking Eq. (22), let us multiply Eq. (24) on the right-hand side by $|\phi\rangle$, that is,

$$[\Box^l - \Box^r]|\Psi\rangle|\phi\rangle = (|\phi\rangle\langle\phi| - |\phi\rangle\langle\phi|)|\phi\rangle = 0$$

Since $\langle \phi | \Box | \phi \rangle = \langle \phi | \phi \rangle m^2$ and as $\langle \phi | \phi \rangle = 1$, we derive the Klein–Gordon equation (23). Using, on the other hand, the bra vector $\langle \phi |$, we obtain the Klein–Gordon equation in the dual Hilbert space obtained from Eq. (24),

$$\langle \phi | \Box | \phi \rangle \langle \phi | - | \phi \rangle \langle \phi | \Box | \phi \rangle = \langle \phi | (m^2 - \Box) = 0$$

This result shows that Eq. (24) is a Liouville–von Neumann equation for the Klein–Gordon field having as the vector state a kind of square root of the density matrix.

This last aspect can be better understood if we write $|\Psi\rangle$ as

$$|\Psi\rangle = \rho^{1/2} |1\rangle$$

with $|1\rangle = \sum_n |n, \tilde{n}\rangle$ [where the usual TFD notation is being used, as in Umezawa (1993)]. Then, Eq. (24) reads

$$[\Box^l — \Box^r]\rho^{1/2} = 0$$

From this equation we have

$$[\Box, \rho] = [P^\mu P_\mu, \rho] = 0 \tag{25}$$

where ρ can be interpreted as the density matrix associated to the Klein–Gordon field, whose general solution is given by $\rho(P^\mu)$. In the following, our main interest is to take Eq. (25) as a starting point to build the relativistic quantum kinetic theory.

Consider that Eq. (25) describes the evolution of an ensemble of quantum particles specified through the density operator ρ, such that the entropy is given by

$$S = -k_B \operatorname{Tr} \rho \ln \rho \tag{26}$$

where k_B is the Boltzmann constant. In the stationary case the entropy is an extremum (see, e.g., Tolman, 1987), that is,

$$\delta S = 0 \tag{27}$$

under the constraints

$$\operatorname{Tr}(\rho) = 1 \tag{28}$$

$$\operatorname{Tr}(\rho N) = \langle N \rangle \tag{29}$$

$$\operatorname{Tr}(\rho P^\nu) = \langle P^\nu \rangle \tag{30}$$

where $\langle N \rangle$, the macroscopic particle number, and $\langle P^\mu \rangle$, the macroscopic four-momentum, are assumed to be constant. Then we obtain as a solution of Eq. (27) that the density operator is

$$\rho_o = \frac{1}{Z} \exp\left[\frac{1}{k_B} (\alpha_\nu P^\nu + \alpha_N N)\right] \tag{31}$$

where

$$Z = \exp\left(1 - \frac{\alpha_z}{k_B}\right) \tag{32}$$

and α_z, α_N, and α_ν are the Lagrange multipliers attached to the constraints (28)–(30), respectively. It should be mentioned that ρ_o given by Eq. (31) is a solution of Eq. (25) assuming that N and P commute with each other.

Using Eq. (31), we derive

$$k_B \ln Z + \alpha_\nu \langle P^\nu \rangle + \alpha_N \langle N \rangle + S = 0$$

With this result, we can obtain a physical interpretation of this approach by a suitable definition of the Lagrange multipliers α_ν and α_z. Thus we can assume that

$$\alpha_\nu = -k_B \beta U_\nu \qquad \text{and} \qquad \alpha_N = k_B \mu \beta$$

where $\beta = 1/k_B T$, T is the temperature at the rest frame, μ is the chemical potential, and U_ν is the macroscopic four-velocity field satisfying the relation $U_\nu U^\nu = 1$. Therefore, Eq. (31) is given by

$$\rho_o = \frac{1}{Z} \exp[-\beta(U_\nu P_\omega - \mu N)] \tag{33}$$

such that the partition function Z is deduced from the normalization of ρ_o.

Now we study the concept of phase space in this approach. Using Eq. (24), we can write Eq. (25) as

$$(\partial^{\mu'} \partial_{\mu'} - \partial^\mu \partial_\mu) \rho(x', x) = 0 \tag{34}$$

Let us introduce the four-operators

$$\frac{\partial}{\partial x^\mu} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial q^\mu} - p_\mu\right) \qquad \text{and} \qquad \frac{\partial}{\partial x'^\mu} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial q^\mu} + p_\mu\right) \tag{35}$$

where q^μ and p_μ, under a Lorentz transformation, transform as a four-position and a four-momentum vector, respectively. In this situation, q^μ and p_μ can be used to introduce the notion of a relativistic phase space. In fact, considering Eq. (35), it is straightforward to show that Eq. (34) is equivalent to

$$p^\mu \frac{\partial}{\partial q^\mu} \rho(q, p) = 0 \tag{36}$$

This equation can be interpreted as a sort of the collisionless Boltzmann equation for the one-particle Wigner distribution $\rho(q, p)$. To establish this clearly, let us explore the physical meaning of $\rho(q, p)$. First note that $\rho(q, p)$ is in principle a Lorentz scalar. Thus an invariant solution of Eq. (36) can be written as

$$\rho(q, p) = \int d^4u \, \delta(p^\nu u_\nu) e^{-u \cdot q} g(p, u) \tag{37}$$

where $g(p, u)$ is an arbitrary function, which can be used either to prevent $\rho(q, p)$ from being divergent, or to specify the microscopic nature of $\rho(q, p)$. For instance, consider $g(p, u)$ in Eq. (37) defined by

$$g(p, u) = \left\langle a\dagger\left(p - \frac{1}{2}u\right) a\left(p + \frac{1}{2}u\right)\right\rangle + \left\langle \bar{a}\dagger\left(p - \frac{1}{2}u\right) \bar{a}\left(p + \frac{1}{2}u\right)\right\rangle \tag{38}$$

where $a(p)$ and $a\dagger(p)$ [$\bar{a}(p)$ and $\bar{a}\dagger(p)$] are the momentum-space operators for bosons (anti-bosons). Therefore, the microscopic specification of the operators N and P^ν in the momentum representation is

$$N = \int \frac{d^3p}{p^o} [a\dagger(p) a(p) + \bar{a}\dagger(p)\bar{a}(p)]$$

$$P^\nu = \int \frac{d^3p}{p^o} p^\nu [a\dagger(p) a(p) + \bar{a}\dagger(p)\bar{a}(p)]$$

Let us define the macroscopic current density by

$$\langle J^\nu \rangle = \int d^4p \, \frac{1}{p^o} p^\mu \rho(q, p). \tag{39}$$

With the operators $a(p)$ and $\bar{a}(p)$, the field operator $\psi(x)$ is written as

$$\psi(x = \frac{1}{2(2\pi)^3} \int \frac{d^3p}{p^o} [e^{-ip\cdot x} a(p) + e^{ip\cdot x} \bar{a}\dagger(p)]$$

Therefore, after some calculation, Eq. (39) reads

$$\langle J^\nu(x) \rangle = i\langle :\psi\dagger(x) \overleftrightarrow{\partial}^\nu \psi(x): \rangle \tag{40}$$

where the dots mean the normal ordering. Equation (40) is the usual definition for the (thermal) current density, and is used as the starting point to introduce the relativistic quantum kinetic theory. Then, in this sense, $\rho(q, p)$ can be interpreted as a one-particle Wigner-function density. Moreover, Eq. (38) is an appropriate choice for $g(p, u)$, providing a physical interpretation for the theory.

Let us go back to the stationary case, and consider $g(p, u)$ in Eq. (37) only for bosons, that is,

$$\begin{aligned} g(p, u) &= \left\langle a\dagger(p - \frac{1}{2} u)\ a(p + \frac{1}{2} u)\right\rangle_o \\ &= \mathrm{Tr}[\rho_o a\dagger(p - \frac{1}{2} u) a(p + \frac{1}{2} u)] \end{aligned} \tag{41}$$

In this way, let us write

$$N = \int \frac{d^3p}{p^o}\, a\dagger(p)a(p)$$

$$P^\mu = \int \frac{d^3p}{p^o}\, p^\mu a\dagger(p)a(p)$$

Then using Eq. (33) in Eq. (41) and the result [see, for instance, de Groot *et al.*, 1980)

$$\begin{aligned} &\left\langle a\dagger\left(p - \frac{1}{2} u\right)a\left(p + \frac{1}{2} u\right)\right\rangle_o \\ &= \left\langle a\left(p + \frac{1}{2} u\right)a\dagger\left(p - \frac{1}{2} u\right)\right\rangle_o \exp(\beta\mu - p^\nu U_\nu) \end{aligned}$$

which is derived from the properties of the trace, we get

$$\rho_o(p) = \frac{1}{\exp[\beta(p^\nu U_\nu - \mu)] - 1}$$

which is the so-called Jüttner distribution.

In short, we have shown how to use the representation of symmetry groups to derive relativistic statistical mechanics. The aspect of simplicity for building up the basis of the relativistic quantum kinetic theory is noteworthy in this formalim. The distribution function $\rho(q, p)$ arises naturally in a covariant form, which is not the case in usual approaches. Finally, we point out that Eq. (36) can be considered as a quantum field equation in phase space, as is the case of the analysis developed by Santana *et al.* (1996) and da Silva *et al.* (1997) for Galilei symmetries. Then we can study, in this context of phase space, a Lagrangian formalism to take into account a nonnull collisionness term in Eq. (36). This analysis, as well as a study of the spinor representations of the *-Poincaré algebra, is in progress.

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